Journal of Engineering Mathematics, Vol. 9, No. 3, July 1975
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Large-time inversion of certain Laplace transforms in dissipative wave propagation

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(Received October 21, 1974)

SUMMARY

A large-time approximation for the inversion of Laplace transforms that commonly occur in dissipative wave propagation is obtained and discussed. This asymptotic approximation properly describes the bulk wave front and satisfies appropriate boundary conditions. The generality of the method is illustrated by means of examples from gas dynamics and viscoelasticity.

1. Introduction

The inversion of Laplace transforms that are used to analyze linearized wave propagation, especially in dissipative media, is often a troublesome proposition. The difficulties can arise from at least three possibilities: (1) In well-known inversion methods such as the steepest-descent method [1], finding the appropriate contour of integration, such as the steepest-descent curve, may not be a trivial exercise. Furthermore, different problems involve individual analyses since the appropriate contours of integration may vary from problem to problem. (2) Although higher-order approximations may be obtained in principle by a suitable expansion in some parameter, the delineation of such a procedure is not always obvious or straightforward. (3) The asymptotic approximation may be valid near a wave front of interest, but it may break down at some other region that is also of interest.

In this paper, for a class of Laplace transforms associated with linearized wave propagation in dissipative media, we obtain an inversion in terms of an asymptotic expansion valid for large times. This expansion properly describes the conditions near the bulk wave front and also satisfies boundary conditions imposed at another point in the disturbance field. Further, the complete asymptotic expansion is delineated. The results of this analysis thus overcome such difficulties as mentioned above and provide a general basis for analyzing a wide range of problems. Even in those cases where exact inversions can be obtained, the inversions may not be in a completely useful form, either for theoretical interpretations or for obtaining numerical results. The results of the present analysis should be of significant value for these cases also.

2. General form of the transforms and examples

We consider Laplace transforms with respect to time of a suitably normalized variable $u(x, t)$ that have the form

$$\bar{u}(x, s) = \bar{G}(x, s)\,\mathrm{e}^{-xF(s)}\,, \tag{1}$$

where s is the Laplace-transform variable and $\bar{G}(x, s)$ is the transform of $G(x, t)$. We assume that x is positive and that $F(s)$ is positive when s is real and positive. The function $F(s)$ is restricted such that it is analytic at $s=0$ and such that $F(0)=0$. We further assume that $F'(0)>0$ and $F''(0)<0$. These restrictions are associated with wave behavior.

Consider several representative examples of expression (1). Consider first the chemical-relaxation flow induced by the impulsive motion of a piston at $x=0$ [2]. In this case u represents the normalized velocity and $\bar{G}=1/s$. The function $F(s)$ is

$$F(s) = \frac{s}{a_f}\left[\frac{\beta^2+\tau_\infty s}{1+\tau_\infty s}\right]^{\frac{1}{2}}, \tag{2}$$

where a_f and a_e are the frozen and equilibrium speeds of sound, $\beta \equiv a_f/a_e$, and τ_∞ is the characteristic relaxation time. For linearized chemically-relaxing Prandtl–Meyer type flow past a sharp corner, Clarke and McChesney [2] show that $F(s)$ has the same form as (2) but $\bar{G}(x, s)$ is more complicated.

As a second example, consider the velocity induced in a one-dimensional viscoelastic medium by an impulsive step velocity at $x=0$ [3]. Again u represents the normalized velocity and $\bar{G}=1/s$, but the function $F(s)$ is given by

$$F(s) = \frac{s}{c}(1+\tau_v s)^{-\frac{1}{2}}, \tag{3}$$

where c is the characteristic dilational speed of sound of the medium and τ_v is a characteristic viscoelastic relaxation time.

As a third example, consider linearized flows with viscous and heat-conduction effects taken into account [4, 5]. In this case, transforms such as (1) occur where $\bar{G}(x, s)$ is a complicated function and $F(s)$ is given by

$$F(s) = \left[\frac{(\gamma+P)s^2+Ps-s\{[(\gamma-P)s+P]^2+4P(P-1)s\}^{\frac{1}{2}}}{2(1+\gamma s)}\right]^{\frac{1}{2}}, \tag{4}$$

where γ is the ratio of specific heats, $P \equiv 4\sigma/3$, and σ is the Prandtl number. When $\sigma=3/4$, expression (4) reduces to the same form as (3). Appropriate boundary conditions are imposed at $x=0$.

The special forms for $F(s)$ shown above are illustrative of the different types of functions that can be considered within the framework of a general analysis. From a steepest-descent analysis, or by the analysis of Whitham [6], one can deduce that the behavior of the bulk-wave front for large time is associated with the value $s=0$. We are therefore interested in the behavior of the function $F(s)$ near $s=0$.

3. Analysis

Since $s=0$ is the pertinent value of s associated with the bulk-wave front, we expand $F(s)$ in a Taylor series about $s=0$:

$$F(s) \sim F_0' s + \tfrac{1}{2}F_0'' s^2 + \tfrac{1}{6}F_0''' s^3 + O(s^4). \tag{5}$$

Here F_0', F_0'', and F_0''' are the first, second, and third derivatives of $F(s)$ evaluated at $s=0$. Keeping only the first two terms of this expansion does not lead to any straightforward approximation that is valid for large time. Among other reasons, the two-term approximation for $F(s)$ will change sign when s becomes as large as $-2F_0'/F_0''$.

We can overcome this difficulty if we expand $F(s)$ in terms of another function, $\lambda(s)$, that has more desirable properties near $s=0$. One such function that will lead to a straightforward inversion is shown below together with its expansion for small s:

$$\lambda(s) \equiv 2ab[(s+b^2)^{\frac{1}{2}}-b]$$

$$\sim as - \frac{a}{4b^2}s^2 + \frac{a}{8b^4}s^3 + O(s^4). \tag{6}$$

We now require that the first two terms of (5) and (6) be the same, and a and b are thus evaluated as

$$a \equiv F_0', \quad b \equiv [-F_0'/2F_0'']^{\frac{1}{2}}. \tag{7}$$

Recall the requirement that $F_0' > 0$ and $F_0'' < 0$. Taking the difference between (5) and (6) now yields

$$F(s) \sim \lambda(s) + \tfrac{1}{6}\left[F_0''' - \frac{3a}{4b^4}\right] s^3 + O(s^4). \tag{8}$$

We can now write, for $s \to 0$,

$$\mathrm{e}^{-xF(s)} \sim \mathrm{e}^{-x\lambda(s)}\left[1 + \sum_{n=3}^{\infty} K_n(x) s^n\right], \tag{9}$$

where

$$K_3(x) \equiv -\frac{x}{6}\left[F_0''' - \frac{3a}{4b^4}\right], \tag{10}$$

and the remaining functions $K_n(x)$ can be obtained by laborious algebra.

The inversion of the first term in (9) is

$$L^{-1}\{\mathrm{e}^{-x\lambda(s)}\} = \frac{abx}{\pi^{\frac{1}{2}}} \frac{\exp\left[-\dfrac{b^2(t-ax)^2}{t}\right]}{t^{\frac{3}{2}}}. \tag{11}$$

By means of the convolution theorem, expression (1) with the substitution of (9) can be inverted term by term. For the leading term in (9) we obtain

$$u_0(x, t) = \frac{abx}{\pi^{\frac{1}{2}}} \int_0^t G(x, t-\tau) \frac{\exp\left[-\dfrac{b^2(\tau-ax)^2}{\tau}\right]}{\tau^{\frac{3}{2}}} d\tau. \tag{12}$$

We now observe that u_0 and all its time derivatives vanish at $t=0$ when $x>0$ (since the function (11) is exponentially small near $t \to 0$ when $x>0$). Thus the higher-order terms associated with s^n in (9) are related to the corresponding nth time derivatives of u_0. The complete asymptotic expansion for large time can thus be written

$$u(x, t) \sim \left[1 + \sum_{n=3}^{\infty} K_n(x) \frac{\partial^n}{\partial t^n}\right] u_0(x, t). \tag{13}$$

Since the functions $K_n(x)$ all vanish at $x=0$, expression (13) is valid at $x=0$ also. The function $u_0(x, t)$ gives the exact value of $u(x, t)$ at $x=0$.

4. Discussion

For $G=1$, corresponding to $u(0, t)=1$ (and pertaining to the chemically-relaxing piston-flow problem and the viscoelastic problem alluded to previously according to (2) and (3)), we obtain from (12)

$$u_0(x, t) = \tfrac{1}{2}\left[\operatorname{erfc}\left\{\frac{b(ax-t)}{t^{\frac{1}{2}}}\right\} + \mathrm{e}^{4ab^2x} \operatorname{erfc}\left\{\frac{b(ax+t)}{t^{\frac{1}{2}}}\right\}\right]. \tag{14}$$

The third time derivative corresponding to the first correction in (13) is

$$u_{0_{ttt}} = t^{-4} u_{0_t}\left[\left\{\frac{5t}{2} + b^2(t^2-a^2x^2)\right\}\left\{\frac{3t}{2} + b^2(t^2-a^2x^2)\right\} - b^2 t(t^2+a^2x^2)\right], \tag{15}$$

where u_{0_t} is given by (11). Expressions (14) and (13) both satisfy the boundary condition $u(0, t)=1$.

Consider now conditions at the bulk wave front, $ax=t$. For large time expression (14) yields

$$u_0\left(\frac{t}{a}, t\right) \sim \frac{1}{2}\left[1 + \frac{1}{2b(\pi t)^{\frac{1}{2}}} + O(t^{-\frac{3}{2}})\right], \tag{16}$$

where exponentially small terms are omitted. Correspondingly, expression (15) yields

$$u_{0_{ttt}}\left(\frac{t}{a}, t\right) \sim -\frac{2b^3}{\pi^{\frac{1}{2}}} \frac{1}{t^{\frac{3}{2}}} + O(t^{-\frac{5}{2}}). \tag{17}$$

The total asymptotic expansion (13) thus behaves as

$$u\left(\frac{t}{a}, t\right) \sim \frac{1}{2}\left[1 + \frac{1}{2b(\pi t)^{\frac{1}{2}}} + O(t^{-\frac{3}{2}})\right]\left[1 + \frac{1}{4b(\pi t)^{\frac{1}{2}}}\left\{\frac{4b^4 F_0'''}{3a} - 1\right\} + O(t^{-\frac{3}{2}})\right]. \quad (18)$$

The terms of order $t^{-\frac{1}{2}}$ can be combined to give the total effect, but it is useful for present purposes to leave expression (18) as it is. The contribution associated with the lowest-order approximation, u_0, is thus portrayed, as is the correction associated with K_3 which depends on the value of F_0'''.

The result that $u = \frac{1}{2}$ at the wave front for large t was pointed out by Clarke and McChesney [2]. In fact, they obtained the first term of expression (14) by a steepest-descent analysis but, because they made the further restriction that $|ax - t|$ was suitably small, they could not satisfy the condition $u(0, t) = 1$ exactly. Also the higher-order corrections were not established.

The large-time approximations above are valid for both problems described by (2) and (3), but the values of a and b are different. For the chemical-relaxation problem associated with (2), we get

$$\begin{aligned} a &= \beta/a_f = a_e^{-1}, \quad b = \frac{\beta}{[2\tau_\infty(\beta^2-1)]^{\frac{1}{2}}}, \\ F_0''' &= \frac{3a\tau_\infty^2}{4\beta^4}(\beta^2-1)(3\beta^2+1), \\ K_3 &= \frac{ax\tau_\infty^2}{8\beta^4}(\beta^2-1)(\beta^2-5). \end{aligned} \quad (19)$$

Notice that the first correction, corresponding to K_3, vanishes when $\beta = 5^{\frac{1}{2}}$. Thus for certain problems, the first correction may vanish when certain parameters take special values, and the first approximation, u_0, is then especially accurate.

For the viscoelastic problem associated with (3) we have

$$a = c^{-1}, \quad b = \frac{1}{(2\tau_v)^{\frac{1}{2}}}, \quad F_0''' = \frac{9a}{16b^4}, \quad K_3 = \frac{ax}{32b^4}. \quad (20)$$

For this problem Morrison [3] has obtained an exact solution, which is actually a quadrature that must be evaluated numerically. It is interesting to compare the present approximation with his results. In Fig. 1, we plot u as a function of $x/c\tau_v$ for $t/\tau_v = 4$ and 6, which represent only moderately large values of time. The first approximation (14) agrees very well with the exact solution. The contribution of the first correction is not shown, but it is small as the evaluation of the second set of brackets in expression (18) shows:

$$u\left(\frac{t}{a}, t\right) \sim \frac{1}{2}\left[1 + \frac{1}{2b(\pi t)^{\frac{1}{2}}} + O(t^{-\frac{3}{2}})\right]\left[1 - \frac{1}{16b(\pi t)^{\frac{1}{2}}} + O(t^{-\frac{3}{2}})\right]. \quad (21)$$

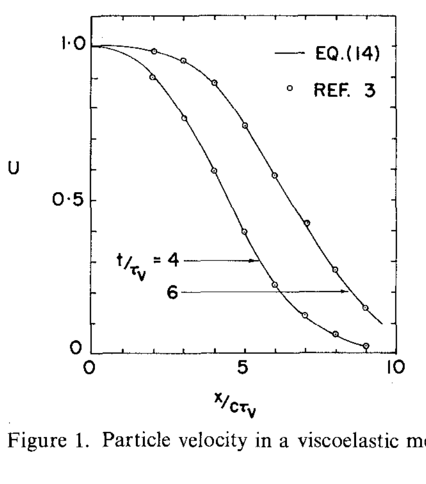

Figure 1. Particle velocity in a viscoelastic medium due to an impulsive velocity at $x = 0$.

The first correction modifies u_0 by only 2 or 3 percent when t/τ_v is as small as 6 or 4. By means of the present formulas this sort of evaluation can always be obtained straightforwardly.

It is interesting to compare the present results with an approximation due to Whitham (see Eq. (48), reference [6]). In terms of our notation, Whitham's formula appears as

$$u(x,t) \simeq \frac{\tilde{b}}{(\pi a x)^{\frac{1}{2}}} \int_0^t G(x, t-\tau) \exp\left[-\frac{\tilde{b}^2(\tau - ax)^2}{ax}\right] d\tau\,, \tag{22}$$

where $\tilde{b}$ corresponds to our parameter b. For the case discussed above where $G=1$, expression (22) integrates to

$$u(x,t) \simeq \frac{1}{2}\left[\operatorname{erfc}\left[-\tilde{b}(ax)^{\frac{1}{2}}\right] - \operatorname{erfc}\left\{\frac{\tilde{b}(t-ax)}{(ax)^{\frac{1}{2}}}\right\}\right]. \tag{23}$$

At the bulk wave front, $ax=t$, this expression yields $u(ta^{-1}, t) \simeq \frac{1}{2}$ plus exponentially small terms, and terms of order $t^{-\frac{1}{2}}$ do not appear. For the conditions shown in Fig. 1, formula (23) leads to a value at the wave front some 16% below the exact value. At $x=0$, formula (23) yields the value $u(0, t) \simeq \frac{1}{2}$ whereas the correct value is $u(0, t)=1$. Thus although formula (23) leads to the correct behavior near the leading part of the bulk wave front for large enough times, it is unsuitable for plotting the complete wave form such as shown in Fig. 1.

5. Concluding remarks

The present analysis produces results that are valid near the bulk wave front and yield the exact value at $x=0$. Furthermore higher-order results are explicitly delineated. The results are useful for theoretical considerations as well as for obtaining numerical results.

When the function $\bar{G}(x, s)$ is not simple, unlike the examples shown, a suitable expansion of $\bar{G}(x, s)$ for small s may be appropriate. In these cases, similar asymptotic representations to that shown above can be obtained. Examples of complicated functions $\bar{G}(x, s)$ that appear in the analysis of weak explosions can be found in references [5] and [7].

In certain problems, matching of two solutions valid on either side of a surface, at $x=0$, say, may be required [5, 7]. When this is true, it is then desirable to preserve the value at $x=0$ as well as at the wave front in an approximate inversion, especially if numerical results are desired for the plotting of profiles and curves. The present method of analysis is especially appropriate for such situations.

REFERENCES

[1] G. F. Carrier, M. Krook and C. E. Pearson, *Functions of a Complex Variable*, McGraw Hill, New York (1966) Chap. 6.
[2] J. F. Clarke and M. McChesney, *The Dynamics of Real Gases*. Butterworths, London (1964) 191–200.
[3] J. A. Morrison, Wave Propagation in Rods of Voigt Material and Viscoelastic Materials with Three-Parameter Models, *Quart. Appl. Math.*, 14 (1956) 153–170.
[4] G. Bienkowsky, Propagation of an Initial Density Discontinuity, *Advances in Applied Mechanics, Rarefied Gas Dynamics*, Vol. 1, Ed. by J. H. DeLeuw, Academic Press, New York (1965).
[5] M. L. Rasmussen and J. G. Lake, Viscous and Heat-Conduction Effects on Weak Spherical Explosions, *Developments in Mechanics*, Vol. 7, Ed. by J. Abrams and T. G. Woo, School of Engineering, University of Pittsburgh (1973) 217–230.
[6] G. B. Whitham, Some Comments on Wave Propagation and Shock-Wave Structure with Application to Magnetohydrodynamics, *Comm. Pure Appl. Math.*, 12 (1959) 113–158.
[7] M. L. Rasmussen, Diffusion Flames Associated with Weak Explosions, Aero. Note 2-74, Cranfield Institute of Technology, College of Aeronautics, Aerodynamics Division (1974).